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NONATTAINABILITY OF A SET BY A DIFFUSION PROCESS

BY

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ABSTRACT. Consider a system of n stochastic differential equations $d\xi = b(\xi)dt + \sigma(\xi)dw$. Let M be a k-dimensional submanifold in R^n, $k \le n-1$. For $x \in M$, denote by $d(x)$ the rank of $\sigma\sigma^*$ restricted to the linear space of all normals to M at x. It is proved that if $d(x) \ge 2$ for all $x \in M$, then $\xi(t)$ does not hit M at finite time, given $\xi(0) \notin M$, i.e., M is non-attainable. The cases $d(x) \ge 1$, $d(x) \ge 0$ are also studied.

Introduction. It is well known that a Brownian motion in n-dimensions, $n \geq 2$, does not hit a prescribed point $x \neq 0$ with probability 1. This result was recently extended by Bonami, Karoui, Roynette and Reinhard [1] to diffusion processes in n-dimension, $n \geq 2$, provided the diffusion matrix is nondegenerate. In another recent paper, Friedman and Pinsky [4] have proved that a diffusion process $\xi(t)$ does not hit a given closed domain Ω, with probability 1, provided $\xi(0) \notin \Omega$ and provided the "normal diffusion" and "normal drift" vanish on $\partial\Omega$.

The purpose of this paper is to prove general theorems of the form

$$P_x\{\xi(t) \in M \text{ for some } t > 0\} = 0 \quad \text{if } x \notin M, \tag{0.1}$$

where $\xi(t)$ is a diffusion process in R^n and M is a manifold in R^n of dimension k, $0 \leq k \leq n-1$. The result in [1] mentioned above will follow as a special case of one of the results (namely, Theorem 4.1) of the present paper.

When (0.1) holds we say that M is *nonattainable* by the process $\xi(t)$.

Denote by $d(x)$ the rank of the diffusion matrix $a(x)$ at x, $x \in M$, when restricted to the linear subspace formed by the normals to M at x. Our results depend in a crucial manner on $d(x)$.

In §1 we give some basic definitions and prove a lemma, which is helpful in "localizing" the proof of nonattainability. In §2 we reduce the problem of estab-

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lishing (0.1) to the problem of finding a solution u of $Lu \leq \mu u$ near M, which "blows up" on M; L is the elliptic operator of the diffusion process and $\mu \geq 0$.

In §3 we establish (0.1) in the case $d \geq 3$. Under somewhat stronger assumptions we establish, in §4, the property (0.1) in case $d \geq 2$.

The cases $d = 1$ and $d = 0$ are dealt with in §§5 and 6 respectively. Finally, in §7, we consider the "mixed" case where $d = 0$ on M, $d = 1$ on ∂M; M is taken to be an arc in R^2. This case is motivated by applications to the Dirichlet problem for degenerate elliptic equations. Thus, it is shown in §7 how recent results of Friedman and Pinsky [5] can be extended by using our result on the "mixed" case.

Results of the type (0.1) when $n = 1$ and M consists of one point can be found in the book of Gikhman and Skorokhod [6] and in [1].

1. Basic definitions. A lemma. Let M be a k-dimensional C^2 manifold in R^n. At each point $x^0 \in M$, let $N^{k+i}(x^0)$ $(1 \leq i \leq n-k)$ form a set of linearly independent vectors in R^n which are normal to M and x^0.

Let $a(x)$ be an $n \times n$ matrix, and consider the $(n-k) \times (n-k)$ matrix $\alpha = (\alpha_{ij})$ where

$$\alpha_{ij} = \langle a(x^0) N^{k+i}(x^0), N^{k+j}(x^0) \rangle \qquad (1 \leq i, j \leq n-k),$$

here $\langle\ ,\ \rangle$ denotes the scalar product in R^n.

Denote the rank of α by $r_{M^\perp}(x^0)$. This number is clearly independent of the choice of the particular set of normals $N^{k+i}(x^0)$.

Definition. The *rank of $a(x)$ orthogonal to M at x^0* is the number $r_{M^\perp}(x^0)$.

If the manifold M has boundary ∂M, then we always take M to be a closed set, i.e., $\overline{M} = M \cup \partial M = M$. If $x^0 \in \partial M$, then by a normal N to M at x^0 we mean a vector N that is $\lim N(x)$, where $x \in \operatorname{int} M$, $x \to x^0$ and $N(x)$ is normal to M at x. We now define $r_{M^\perp}(x^0)$, for $x^0 \in \partial M$, in the same way as before.

Notice that ∂M is also a manifold, and one can define $r_{(\partial M)^\perp}(x^0)$. Clearly, $r_{(\partial M)^\perp}(x^0) \geq r_{M^\perp}(x^0)$.

Notice also that when M consists of just one point x^0, $r_{M^\perp}(x^0)$ is the rank of the matrix $a(x^0)$.

Consider now a diffusion process governed by a system of n stochastic differential equations

$$d\xi(t) = \sigma(\xi(t))\,dw + b(\xi(t))\,dt; \tag{1.1}$$

$\sigma(x)$ is an $n \times n$ matrix $(\sigma_{ij}(x))$, $b(x)$ is a column vector $(b_1(x), \cdots, b_n(x))$, and $w(t)$ is an n-dimensional Brownian motion $(w^1(t), \cdots, w^n(t))$.

We assume:

(A) $\sigma(x)$ and $b(x)$ satisfy, for all $x \in R^n$,

$$|\sigma(x)| + |b(x)| \leq C(1 + |x|) \quad (C \text{ constant});$$

further, for any $R > 0$ there is a positive constant C_R such that $|\sigma(x) - \sigma(y)| + |b(x) - b(y)| \leq C_R|x - y|$ if $|x| < R$, $|y| < R$.

Introduce the diffusion matrix $a(x) = (a_{ij}(x))$:

$$a(x) = \tfrac{1}{2}\sigma(x)\sigma^*(x) \qquad [\sigma^*(x) = \text{transpose of } \sigma(x)],$$

and denote the rank of $a(x)$ orthogonal to M at x by $d(x)$, i.e.,

$$d(x) = r_{M^\perp}(x) \quad \text{for } x \in M. \tag{1.2}$$

Definition. A closed set M in R^n is *nonattainable* by the process $\xi(t)$ if

$$P_x\{\xi(t) \in M \text{ for some } t > 0\} = 0 \quad \text{for each } x \notin M. \tag{1.3}$$

It will be shown later on that if $d(x) \geq 2$ for all $x \in M$ (M a C^2 manifold) then M is nonattainable. The same assertion is true in some cases when $d(x) \geq 1$ (but not always), provided $n \geq 2$. The interpretation of these results is that M is "too thin" for $\xi(t)$ to hit it.

It will also be shown that when $d(x) \equiv 0$ on M, then the assertion (1.3) is still true provided the "normal drift" of $\xi(t)$ vanishes on M. The interpretation of this result is that M is an "obstacle" for the diffusion process $\xi(t)$.

We conclude this section with a lemma that will be useful in reducing the proof of the assertion (1.3) from a global manifold M to a local one.

Let $x^0 \in M$. Then, in a neighborhood of x^0, M can be represented in the form

$$x_{i'} = f_{i'}(x'') \tag{1.4}$$

where i' varies over $n - k$ of the indices $1, 2, \cdots, n$, the coordinates of x'' are $x_{i''}$, and i'' varies over the remaining indices. Suppose for simplicity that i' varies over $k + 1, \cdots, n$, i.e., M is given locally by

$$x_{k+i} = f_{k+i}(x_1, \cdots, x_k) \qquad (i = 1, \cdots, n - k). \tag{1.5}$$

Introduce the mapping

$$\begin{aligned} y_i &= x_i - x_i^0 && (i = 1, \cdots, k), \\ y_{k+i} &= x_{k+i} - f_{k+i}(x_1, \cdots, x_k) && (i = 1, \cdots, n - k), \end{aligned} \tag{1.6}$$

where $x^0 = (x_1^0, \cdots, x_n^0)$. This is a diffeomorphism from a neighborhood $V(x^0)$ of x^0 into a neighborhood V^* of 0 in the y-space. Denote by M^* the image of $M \cap V(x^0)$. Then M^* is given by

(1.7) $$y_i = 0 \quad (i = 1, \cdots, k), (y_{k+1}, \cdots, y_n) \in A,$$

for some set A.

Consider the elliptic operator

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i}$$

and set $v(y) = u(x)$. Then $Lu(x) = L'v(y)$ where

$$L'v = \sum_{i,j=1}^{n} a_{ij}^*(y) \frac{\partial^2 v}{\partial y_i \partial y_j} + \sum_{i=1}^{n} b_i^*(y) \frac{\partial v}{\partial y_i}.$$

It is easily seen that $a_{k+i,k+j}^*(y) = \langle a(x)N^{k+i}(x), N^{k+j}(x)\rangle$ where

$$N^{k+i}(x) = \nabla_x g_{k+i}(x), \qquad g_{k+i}(x) = x_{k+i} - f_{k+i}(x_1, \cdots, x_k).$$

Notice that if $x \in M \cap V(x^0)$ then the $N^{k+i}(x)$ $(1 \leq i \leq n-k)$ form a set of linearly independent normal vectors to M at x. Hence

(1.8) $$d(x) = \operatorname{rank}(a_{k+i,k+j}^*(x))_{i,j=1}^{n-k} \qquad (x \in M \cap V(x^0)).$$

By performing an affine transformation in the space of variables $(y_{k+1}, \cdots, y_n)$ we do not affect the manifold M^* given by (1.7), except for a change in the set A. At the same time, after performing such a transformation we can achieve the conditions

(1.9) $$\hat{a}_{k+i,k+j}(0) = \begin{cases} 1 & \text{if } i = j = k+1, \cdots, d(x^0), \\ 0 & \text{for all other } i, j \ (1 \leq i, j \leq n-k), \end{cases}$$

where $\hat{a}_{k+i,k+j}$ are the new $a_{k+i,k+j}^*$.

Next, by an affine transformation in the space of variables $(y_1, \cdots, y_k)$ we do not affect the manifold M^*. At the same time we can achieve the additional conditions

(1.10) $$\tilde{a}_{i,j}(0) = \begin{cases} \eta & \text{if } i = j = 1, \cdots, d^* \ (\eta > 0), \\ 0 & \text{for all other } i, j \ (1 \leq i, j \leq k), \end{cases}$$

where η is any given positive number, d^* is the rank of the matrix $(\hat{a}_{ij}(0))_{i,j=1}^{k}$ and $\tilde{a}_{i,j}$ are the new $\hat{a}_{i,j}$. Notice that d^* can be any number ≥ 0 and $\leq k$.

Notation. Let B be any set in R^n and let $x \in R^n$. The distance from x to B will be denoted by $d(x, B)$.

Let $M_V = M \cap V(x^0)$. Let W be a neighborhood of M_V. We shall be interested, later on, in finding a function u satisfying:

$$(1.11)\qquad \begin{aligned} &Lu(x) \le \mu u(x) \ \text{ if } x \in W\setminus M_V \quad (\mu \text{ nonnegative constant}),\\ &u(x) \to \infty \quad \text{if } x \in W\setminus M_V,\ d(x, M_V) \to 0. \end{aligned}$$

Suppose after performing the transformation (1.6) and the two affine transformations used above (to get (1.9), (1.10)), we can construct a function $u'(x')$ satisfying (1.11) in the new x'-variable and with the transformed L and M. Then the function $u(x) = u'(x')$ will satisfy (1.11). Consequently, in trying to prove the existence of $u(x)$ satisfying (1.11), we may, without loss of generality, assume that M is given by

$$(1.12)\qquad x_{k+1} = 0, \cdots, x_n = 0,$$

that $x^0 = 0$, and that

$$(1.13)\qquad a_{k+i,k+j}(0) = \begin{cases} 1 & \text{if } i = j = 1, \cdots, d(0),\\ 0 & \text{for all other } i, j;\ 1 \le i, j \le n - k, \end{cases}$$

$$(1.14)\qquad a_{i,j}(0) = \begin{cases} \eta & \text{if } i = j = 1, \cdots, d^* \ (\eta > 0),\\ 0 & \text{for all other } i, j;\ 1 \le i, j \le k, \end{cases}$$

for some $0 \le d^* \le k$.

In the above arguments we have assumed the local representation (1.5). The same arguments apply, of course, also in the general case where M has a local representation of the form (1.4). We sum up:

Lemma 1.1. *In order to find a function u satisfying* (1.11), *we may assume, without loss of generality, that $x^0 = 0$, that M is given by* (1.12) *and that* (1.13), (1.14) *hold.*

The following result can be obtained by slightly modifying the proof of Lemma 1.1.

Lemma 1.1′. *Let p be a given positive number. In order to find a function u satisfying $Lu(x) \le -\mu/(d(x, M))^p$ if $x \in W\setminus M_V$ (μ positive constant), $u(x) \to \infty$ if $x \in W\setminus M_V$, $d(x, M_V) \to 0$, we may assume, without loss of generality, that $x^0 = 0$, that M is given by* (1.12) *and that* (1.13), (1.14) *hold.*

2. **A fundamental lemma.** *A function $v(x)$ is said to be piecewise continuous in a region G of R^n if there is in G a finite number of C^1 hypersurfaces $S_1, \cdots, S_l$*

and a finite number of C^1 manifolds of dimensions $\leq n-2$, $V_1, \cdots, V_h$, such that:

(i) *for any compact subset G_0 of G, $v(x)$ is continuous and bounded on the set $G_0 \setminus (S \cup V)$ where $S = \bigcup_{i=1}^{l} S_i$, $V = \bigcup_{i=1}^{h} V_i$, and*

(ii) *$v(x)$ $(x \in G \setminus (S \cup V))$ tends to a limit from either side of each S_i.*

Notation. The gradient of v is denoted by $D_x v$. The gradient of $D_x v$ is denoted by $D_x^2 v$.

Let Ω be an open set in R^n. Denote by $\partial\Omega$ the boundary of Ω, and by $\overline{\Omega}$ the closure of Ω. Let

$$\tau = \text{exit time of } \xi(t) \text{ from } \Omega.$$

Let K be a compact subset of $\overline{\Omega}$. For any $\epsilon > 0$, let

$$K_\epsilon = \{x \in \Omega;\ d(x, K) \leq \epsilon\}, \qquad \hat{K}_\epsilon = K_\epsilon \setminus K.$$

Notice that K need not lie entirely in Ω, i.e., $K \cap \partial\Omega$ may be nonempty. The following lemma will be fundamental for the subsequent developments.

Lemma 2.1. *Let* (A) *hold. Let u be a continuously differentiable function in $\hat{K}_{\epsilon_0}$, for some $\epsilon_0 > 0$, and let $D_x^2 u$ be piecewise continuous in $\hat{K}_{\epsilon_0}$. Denote by $S_1, \cdots, S_l$ the $(n-1)$-dimensional manifolds of discontinuity of $D_x^2 u$, and by $V_1, \cdots, V_h$ the manifolds of discontinuity of $D_x^2 u$ of dimensions $\leq n-2$. Let $S = \bigcup_{i=1}^{l} S_i$, $V = \bigcup_{i=1}^{h} V_i$. Suppose*

$$Lu(x) \leq \mu u(x) \quad \text{if } x \in \hat{K}_{\epsilon_0} \setminus (S \cup V) \ (\mu \text{ positive constant}), \tag{2.1}$$

$$u(x) \to \infty \text{ if } x \in \hat{K}_{\epsilon_0}, \quad d(x, K) \to 0. \tag{2.2}$$

Then, for any $x \in \Omega \setminus K$,

$$P_x\{\xi(t) \in K \text{ for some } 0 \leq t < \tau\} = 0. \tag{2.3}$$

This lemma was implicitly stated and proved in [3], [4] in the special case where K is a point or a bounded closed domain, $\Omega = R^n$, $\hat{K}_{\epsilon_0}$ is replaced by R^n, and u is twice continuously differentiable in $R^n \setminus K$.

Proof. Let R, ρ be positive numbers; R will be arbitrarily large and ρ arbitrarily small. Set

$$B_R = \{x;\ |x| < R\}, \qquad \Omega_\rho = \{x \in \Omega;\ d(x, \partial\Omega) > \rho\}.$$

R is such that $K \subset B_R$.

Fix a number ϵ_1, $0 < \epsilon_1 < \epsilon_0$ and let $0 < \epsilon' < \epsilon < \epsilon_1$.

Modify and extend u inside $K_{\epsilon'}$ and outside K_{ϵ_1} so as to obtain a function U in Ω satisfying:

$$\text{(2.4)} \qquad \begin{aligned} &U \text{ and } D_x U \text{ are continuous in } \Omega;\\ &D_x^2 U \text{ is piecewise continuous in } \Omega;\\ &U \text{ is positive in } \Omega. \end{aligned}$$

Since (by (2.2)) $u(x)$ is positive in some Ω-neighborhood of K, we can accomplish (2.4) provided ϵ_1 is sufficiently small.

Denote by Σ the set of discontinuities of $D_x^2 U$. Clearly, for any ρ, R,

$$\text{(2.5)} \qquad \begin{aligned} |D_x U| + |D_x^2 U| &\le C(\rho, R) \quad \text{if } x \in (\Omega_{\rho/2} \setminus K_{\epsilon_1}) \cap B_{R+1},\ x \notin \Sigma,\\ U &\ge c(\rho, R) \quad \text{if } x \in (\Omega_{\rho/2} \setminus K_{\epsilon_1}) \cap B_{R+1}, \end{aligned}$$

where $C(\rho, R)$, $c(\rho, R)$ are positive constants depending on ρ, R, but independent of ϵ'. Since $U = u$ in $K_{\epsilon_1} \setminus K_{\epsilon'}$, we conclude, upon using (2.1) and (2.5), that

$$\text{(2.6)} \qquad LU(x) \le \mu_{\rho,R} U(x) \quad \text{if } x \in (\Omega_{\rho/2} \setminus K_{\epsilon'}) \cap B_{R+1},\ x \notin \Sigma,$$

where $\mu_{\rho,R}$ is a positive constant depending on ρ, R, but independent of ϵ'.

Let $p(x)$ be a C^∞ function in R^n, with support in the unit ball $|x| \le 1$, such that $p(x) \ge 0$, $\int_{R^n} p(x)dx = 1$. For any $\lambda > 0$, we introduce the mollifier $U_\lambda(x)$ of $U(x)$ defined by (cf. [2])

$$\text{(2.7)} \qquad U_\lambda(x) = \int_{|y-x|<\lambda} U(y) p_\lambda(x-y)\,dy \qquad [p_\lambda(x) = (1/\lambda^n) p(x/\lambda)].$$

We take $\lambda < \rho/2$, $\lambda < \epsilon - \epsilon'$, $x \in \Omega_\rho$. Then $U_\lambda(x)$ is in $C^\infty(\Omega_\rho)$, and

$$\text{(2.8)} \qquad D_x U_\lambda(x) = \int_{|y-x|<\lambda} D_y U(y) \cdot p_\lambda(x-y)\,dy.$$

Also,

$$\text{(2.9)} \qquad D_x^2 U_\lambda(x) = -\int_{|y-x|<\lambda} D_y U(y) \cdot D_y p_\lambda(x-y)\,dy.$$

If $d(x, \Sigma) > \lambda$ then clearly

$$D_x^2 U_\lambda(x) = \int_{|y-x|<\lambda} D_y^2 U(y) \cdot p_\lambda(x-y)\,dy. \tag{2.10}$$

Suppose next that $d(x, \Sigma) \le \lambda$ and $\Sigma \cap \{y;\ |y-x| \le \lambda\}$ consists of a hypersurface S_1. Then S_1 divides $\{y;\ |y-x| \le \lambda\}$ into two sets: $S_{1\lambda}$ and $S_{2\lambda}$. Integrating by parts in (2.9) over $S_{1\lambda}$ and $S_{2\lambda}$ separately, and using the continuity of $D_y U$ across S_1, we again get (2.10).

If $\Sigma \cap \{y;\ |y-x| \le \lambda\}$ consists of manifold V of dimension $\le n-2$, then we surround V by an η-neighborhood V_η, and split the integral in (2.9) into a part I_1 integrated over $\{y;\ |y-x| < \lambda\} \cap V_\eta$ and a part I_2. In I_2 we integrate by parts so as to obtain

$$I_2 = \int_{W_\eta} D_y^2 U(y) \cdot p_\lambda(x-y)\,dy + O(\eta), \qquad W_\eta = \{y;\ |y-x| < \lambda,\ y \notin V_\eta\}.$$

Taking $\eta \to 0$ in $I_1 + I_2$, (2.10) follows.

Finally, the general case where $d(x, \Sigma) \le \lambda$ can be handled by combining the above two special cases. Thus (2.10) holds in general.

From (2.7), (2.8), and (2.10) we obtain

$$LU_\lambda(x) - \mu_{\rho,R} U_\lambda(x) = \int_{|y-x|<\lambda} [LU(y) - \mu_{\rho,R} U(y)] p_\lambda(x-y)\,dy.$$

Using (2.6), we get

$$LU_\lambda(x) \le \mu_{\rho,R} U_\lambda(x) \quad \text{if } x \in (\Omega_\rho \backslash K_\epsilon) \cap B_R. \tag{2.11}$$

Let $\tau^0 = \tau_{\rho,R,\epsilon}$ = exit time of $\xi(t)$ from $(\Omega_\rho \backslash K_\epsilon) \cap B_R$, and write, for simplicity, $\mu = \mu_{\rho,R}$. By Itô's formula, if $x \in (\Omega_\rho \backslash K_\epsilon) \cap B_R$, $T > 0$, then

$$\begin{aligned} &E_x\{e^{-\mu(\tau^0 \wedge T)} U_\lambda(\xi(\tau^0 \wedge T))\} - U_\lambda(x) \\ &\qquad = E_x \int_0^{\tau^0 \wedge T} e^{-\mu s}(L-\mu) U_\lambda(\xi(s))\,ds. \end{aligned} \tag{2.12}$$

Notice that $\xi(s) \in (\Omega_\rho \backslash K_\epsilon) \cap B_R$ if $0 \le s < \tau^0 \wedge T$. Hence, by (2.11), the integral on the right-hand side is ≤ 0. Taking $\lambda \to 0$ in (2.12) and using the fact that $U_\lambda(y) \to U(y)$ uniformly in $y \in (\Omega_\rho \backslash K_\epsilon) \cap B_R$, we get

$$E_x\{e^{-\mu(\tau^0 \wedge T)} U(\xi(\tau^0 \wedge T))\} - U(x) \le 0.$$

Since $U > 0$, this yields

$$E_x\{e^{-\mu(\tau^0\wedge T)}U(\xi(\tau^0\wedge T))I_{\{\xi(\tau^0\wedge T)\in\partial K_{\epsilon,\rho}\}}\}\leq U(x) \tag{2.13}$$

where $\mu=\mu_{\rho,R}$, $\tau^0=\tau_{\rho,R,\epsilon}$, $\partial K_{\epsilon,\rho}=\partial K_\epsilon\cap\Omega_\rho$, and ∂K_ϵ is the boundary of K_ϵ; here I_A is the indicator function of a set A.

Noting that $U(\xi(\tau^0\wedge T))\geq\inf_{\partial K_\epsilon\cap\Omega}u(y)$ if $\xi(\tau^0\wedge T)\in\partial K_{\epsilon,\rho}$, and taking $T\to\infty$ in (2.13), we get

$$E_x\{e^{-\mu\tau^0}I_{\{\tau^0<\infty\}}I_{\{\xi(\tau^0)\in\partial K_{\epsilon,\rho}\}}\}\leq U(x)\Big/\Big[\inf_{y\in\partial K_\epsilon\cap\Omega}u(y)\Big]. \tag{2.14}$$

Suppose now that the assertion (2.3) is false. Then there exists a set G of positive probability such that: if $\omega\in G$ then $\xi(t,\omega)\in K$ for some finite $t=t^*(\omega)<\tau(\omega)$. This implies that for all small ϵ, say $0<\epsilon<\epsilon^*$, $\xi(s,\omega)\in\Omega_\rho\cap B_R$ if $0\leq s\leq t_\epsilon$ for some small $\rho>0$ and large R, $\xi(s,\omega)\notin K_\epsilon$ if $0\leq s<t_\epsilon$, and $\xi(t_\epsilon,\omega)\in K_\epsilon$; here $t_\epsilon=t_\epsilon(\omega)\leq t^*(\omega)$, ρ and R are independent of ϵ (but they depend on ω) and one can take, for instance, $\epsilon^*=\epsilon_1$ where ϵ_1 is as above.

Setting $\rho_m=1/m$, $R_m=m$,

$$G_m=G\cap\{\tau_{\rho_m,R_m,\epsilon}<\infty;\ \xi(\tau_{\rho_m,R_m,\epsilon})\in\partial K_{\epsilon,\rho_m}\quad\text{for all }0<\epsilon<\epsilon^*\}$$

we then have: $G=\bigcup_{m=1}^\infty G_m$. Since $P_x(G)>0$, it follows that $P_x(G_m)>0$ for some m. If we take $\rho=\rho_m$, $R=R_m$ in (2.14), and let $\epsilon\to 0$, we obtain, after using (2.2),

$$E_x\{\exp[-\mu_{\rho_m,R_m}\tau_{\rho_m,R_m,\epsilon}]\cdot I_{G_m}\}\to 0\quad\text{if }\epsilon\to 0.$$

This implies that for almost all $\omega\in G_m$,

$$\tau_{\rho_m,R_m,\epsilon}(\omega)\to\infty\quad\text{if }\epsilon\to 0. \tag{2.15}$$

But if $\omega\in G_m$ then $\tau_{\rho_m,R_m,\epsilon}(\omega)\leq t^*(\omega)<\infty$, which contradicts (2.15), since $P_x(G_m)>0$.

Remark. The above proof remains valid in case u is continuous in $\hat{K}_{\epsilon_0}$ and has two strong derivatives in $L^2(A)$ for any compact subset A of $\hat{K}_{\epsilon_0}$, (2.1) holds almost everywhere, and (2.2) holds. Indeed, the assertions (2.8), (2.10) are then valid by definition of strong derivatives (see [2]), and the rest of the proof is essentially the same.

3. **The case $d(x)\geq 3$.** When we speak of a manifold M with boundary ∂M, it is always assumed that M is a closed set, i.e., $\partial M\subset M$.

Theorem 3.1. *Let M be a k-dimensional C^2 submanifold of R^n $(0\leq k\leq n-1)$*

with C^2 boundary ∂M (∂M may be empty), and let (A) *hold. Suppose $d(x) \geq 3$ for each $x \in M$. Then* (1.3) *holds, i.e., M is nonattainable.*

Proof. If the assertion is not true then for some $x \notin M$ there is a point $x^0 \in M$ such that, for any $\delta_0 > 0$,

$$P_x\{\xi(t) \in M \cap B_{\delta_0} \text{ for some } t > 0\} > 0 \tag{3.1}$$

where B_{δ_0} is the closed ball with center x^0 and radius δ_0.

Consider first the case where $x^0 \notin \partial M$. We want to apply Lemma 2.1 with $\Omega = R^n$, $K = M \cap B_{\delta_0}$. Thus we wish to construct a function u in a δ-neighborhood W_δ of K such that

$$\begin{aligned} Lu(x) &\leq \mu u(x) \quad \text{if } x \in W_\delta \setminus K \ (\mu \geq 0), \\ u(x) &\to \infty \qquad \text{if } x \in W_\delta \setminus K,\ d(x, K) \to 0. \end{aligned} \tag{3.2}$$

In view of Lemma 1.1, we may assume that $x^0 = 0$,

$$K = \{x;\ x_{k+1} = 0, \cdots, x_n = 0,\ (x_1, \cdots, x_k) \in A\} \tag{3.3}$$

and that the $a_{ij}(x)$ satisfy (1.13), (1.14) with a given arbitrarily small $\eta > 0$. Further, since δ_0 can be taken arbitrarily small, we may assume that A is a k-dimensional cube, say

$$A = A_\epsilon = \{(x_1, \cdots, x_k);\ -\epsilon \leq x_i \leq \epsilon \text{ for } i = 1, \cdots, k\} \tag{3.4}$$

and ϵ is sufficiently small. We shall determine later on how small ϵ and η are going to be. Also δ can be taken arbitrarily small.

Set $x = (x', x'')$ where $x' = (x_1, \cdots, x_k)$, $x'' = (x_{k+1}, \cdots, x_n)$, and let $r = r(x) = |x''|$. Thus $r(x)$ is the distance from x to K provided $x' \in A_\epsilon$.

Let

$$u(x) = \phi(r) = \log r^{-1} \quad \text{if } x \in W_\delta \setminus K,\ x' \in A_\epsilon. \tag{3.5}$$

Then $u_{x_i} = -x_i/r^2$, $u_{x_i x_j} = -\delta_{ij}/r^2 + 2(x_i x_j/r^2)$ if $k + 1 \leq i, j \leq n$, and $u_{x_i x_j} = 0$ otherwise. Hence, if $d = d(0)$,

$$\sum_{i=1}^{d} a_{k+i,k+i}(0) \frac{\partial^2 u}{\partial x_{k+i}^2} = -\frac{d}{r^2} + 2\,\frac{x_{k+1}^2 + \cdots + x_{k+d}^2}{r^4} \leq -\frac{1}{r^2}$$

since $d \geq 3$. If $i = j > d$ or if $i \neq j$, $k + 1 \leq i, j \leq n$, then

$$\left| a_{k+i,k+j}(x) \frac{\partial^2 u}{\partial x_{k+i} \partial x_{k+j}} \right| = |a_{k+i,k+j}(x) - a_{k+i,k+j}(0)| \left| \frac{\partial^2 u}{\partial x_{k+i} \partial x_{k+j}} \right|$$

$$\leq C|x|/r^2 \leq C(\delta + \epsilon)/r^2$$

where C is a generic constant. Also

$$\left| [a_{k+i,k+i}(x) - a_{k+i,k+i}(0)] \frac{\partial^2 u}{\partial x_{k+i}^2} \right| \leq \frac{C(\delta + \epsilon)}{r^2} \quad \text{if } 1 \leq i \leq d.$$

Noting also that $a_{ij} u_{x_i x_j} = 0$ if either $1 \leq i \leq k$ or $1 \leq j \leq k$, and that $|b_i u_{x_i}| \leq C|u_{x_i}| \leq C/r$, we conclude that

$$Lu \leq -\frac{1}{r^2} + \frac{C(\delta + \epsilon)}{r^2} < -\frac{1}{2r^2} \quad \text{if } x \in W_\delta \backslash K,\ x' \in A_\epsilon$$

provided $\delta + \epsilon < 1/(2C)$.

We next extend the definition of $u(x)$ to the set of points (x', x'') in $W_\delta \backslash K$ where $x' \notin A_\epsilon$. We begin with the subset where

$$x_1 > \epsilon, \quad -\epsilon \leq x_i \leq \epsilon \quad \text{if } 2 \leq i \leq k. \tag{3.6}$$

Let $r_1 = r_1(x) = \{(x_1 - \epsilon)^2 + |x''|^2\}^{1/2}$ if $x \in W_\delta \backslash K$, x' satisfies (3.6). Thus $r_1(x)$ is the distance from x to K. Define $u(x) = \log 1/r_1$ if $x \in W_\delta \backslash K$ and x' satisfies (3.6).

Denote by L' the operator L when $a_{11}(x)$ and the $a_{1,k+i}(x)$, $a_{k+i,1}(x)$ $(1 \leq i \leq d)$ are replaced by 0. Then, by the same calculation as before,

$$L'u(x) < -1/2r_1^2. \tag{3.7}$$

Since $a_{11}(0) = \eta$, $a_{11}(x) < \eta + C(\delta + \epsilon)$ if $x \in W_\delta$. Recalling that $a(x)$ is a positive semidefinite matrix, we also have

$$|a_{1,k+i}(x)| \leq \sqrt{a_{11}(x)}\ \sqrt{a_{k+i,k+i}(x)} \leq C(\eta + \delta + \epsilon)^{1/2}\ (x \in W_\delta).$$

Since $|\partial^2 u/\partial x_1 \partial x_j| \leq 3/r_1^2$, we conclude that $|Lu - L'u| \leq 6nC(\eta + \delta + \epsilon)^{1/2}/r_1^2$ ($x \in W_\delta$, x' satisfies (3.6)). Combining this with (3.7) and taking η (and δ, ϵ) to be sufficiently small, we get $Lu(x) < -1/3r_1^2$ if $x \in W_\delta \backslash K$, x' satisfies (3.6).

Notice that r^2 and r_1^2 agree with their first derivatives on the set where $x_1 = \epsilon$. Hence the function $u(x)$ constructed so far is continuously differentiable, and $D_x^2 u$ is piecewise continuous.

Similarly we extend the definition of $u(x)$ to each of the subsets M_i, N_i $(1 \leq i \leq k)$ of $W_\delta \backslash K$ given by

$$M_i = \{x \in W_\delta \setminus K,\ x_i > \epsilon,\ -\epsilon \le x_j \le \epsilon \text{ if } 1 \le j \le k,\ j \ne i\},$$

$$N_i = \{x \in W_\delta \setminus K,\ x_i < -\epsilon,\ -\epsilon \le x_j \le \epsilon \text{ if } 1 \le j \le k,\ j \ne i\}.$$

Next we extend the definition of $u(x)$ to the subset Γ of $W_\delta \setminus K$ where $x_1 > \epsilon$, $x_2 > \epsilon$. Introducing

$$r_{12}(x) = \{(x_1 - \epsilon)^2 + (x_2 - \epsilon)^2 + |x''|^2\}^{1/2},$$

we define $u(x) = \log(1/r_{12}(x))$. Again we have (if η, δ, ϵ are sufficiently small) $Lu < -c/(r_{12})^2$ for some positive constant c. Notice that the functions r_{12}^2, r_1^2 and their first derivatives agree on the set $x_2 = \epsilon$. Similarly the functions r_{12}^2 and $r_2^2 = (x_2 - \epsilon)^2 + |x''|^2$ and their first derivatives agree on the set $x_1 = \epsilon$. Hence, the function $u(x)$ constructed so far is continuously differentiable, and $D_x^2 u$ is piecewise continuous.

We extend the definition of u, in a similar manner, to the subsets of $W_\delta \setminus K$ defined by $x_i > \epsilon, x_j > \epsilon$, or $x_i > \epsilon$, $x_j < -\epsilon$, or $x_i < -\epsilon$, $x_j < -\epsilon$, for some $i \ne j$, $1 \le i, j \le k$. Then we proceed to define $u(x)$ on sets determined by three inequalities, i.e., $x_i > \epsilon$ or $x_i < -\epsilon$, $x_j > \epsilon$ or $x_j < -\epsilon$, $x_h > \epsilon$ or $x_h < -\epsilon$; etc. The resulting function $u(x)$ is continuously differentiable in the entire set $W_\delta \setminus K$, $D_x^2 u$ is piecewise continuous, and $Lu(x) < 0$ at all the points of $W_\delta \setminus K$ where $D_x^2 u$ exists. Finally, it is clear that $u(x) \to \infty$ if $x \in W_\delta \setminus K$, $d(x, K) \to 0$.

Having constructed u which satisfies (3.2) in the special case where (3.3) and (1.13), (1.14) hold, we appeal to Lemma 1.1 in order to conclude the existence of a continuously differentiable function u, with $D_x^2 u$ piecewise continuous, which satisfies (3.2) in the general case where $K = M \cap B_{\delta_0}$. Applying Lemma 2.1, it follows that

$$P_x\{\xi(t) \in K \text{ for some } t > 0\} = 0 \quad \text{for any } x \notin K.$$

This, however, contradicts (3.1).

We have assumed so far that $x^0 \notin \partial M$. If $x^0 \in \partial M$ then the proof is similar. The set A_ϵ is simply replaced by its intersection with the half space $x_1 \ge 0$.

4. The case $d(x) \ge 2$. We first consider the case where M consists of one point x^0. The number $d(x^0)$ now means the rank of the matrix $a(x^0)$.

Theorem 4.1. *Let* (A) *hold and let* $d(x^0) \ge 2$. *Then*

$$P_x\{\xi(t) = x^0 \text{ for some } t > 0\} = 0 \quad \text{for any } x \ne x^0. \tag{4.1}$$

Proof. We may take $x^0 = 0$. We wish to construct a function u such that

$$Lu(x) \le 0 \quad \text{if } 0 < |x| < \delta, \tag{4.2}$$

(4.3) $$u(x) \to \infty \quad \text{if } |x| \to 0,$$

where δ is a sufficiently small positive number, and $u(x)$ is in C^2 for $0 < |x| < \delta$. In view of Lemma 2.1, this will complete the proof of (4.1).

Because of Lemma 1.1, we may assume, without loss of generality, that

(4.4) $$a_{ii}(0) = 1 \quad \text{if } i = 1, \cdots, d,$$
$$a_{ij}(0) = 0 \quad \text{if } i = j > d \text{ or if } i \neq j.$$

We shall take $u(x) = \phi(r)$ where $r = |x|$ and where $\phi(r)$ is defined by

(4.5) $$\phi'(r) = -e^{r^\theta/\theta}/r, \qquad \phi(0) = \infty,$$

for some constant θ, $0 < \theta < 1$. Since (4.3) clearly holds, it remains to verify (4.2). Now,

$$u_{x_i} = -\frac{x_i}{r^2} e^{r^\theta/\theta}, \qquad u_{x_i x_j} = \left[-\frac{\delta_{ij}}{r^2} + 2\frac{x_i x_j}{r^4} - \frac{x_i x_j}{r^4} r^\theta \right] e^{r^\theta/\theta}.$$

Using the fact that $d \geq 2$, we get

$$\sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} = \left[-\frac{d}{r^2} + 2\frac{x_1^2 + \cdots + x_d^2}{r^4} - \frac{x_1^2 + \cdots + x_d^2}{r^4} r^\theta \right] e^{r^\theta/\theta}$$
$$\leq \left[-2\frac{x_{d+1}^2 + \cdots + x_n^2}{r^4} - \frac{x_1^2 + \cdots + x_d^2}{r^4} r^\theta \right] e^{r^\theta/\theta} \leq -\frac{r^\theta}{r^2}$$

if $r < 1$. On the other hand,

$$|[a_{ij}(x) - a_{ij}(0)] u_{x_i x_j}| \leq C|x| \frac{1}{r^2} \leq \frac{C}{r}, \qquad |b_i(x) u_{x_i}| \leq C|u_{x_i}| \leq \frac{C}{r}.$$

Recalling (4.4), we conclude that $Lu \leq -r^\theta/r^2 + C/r < 0$ if $0 < r < \delta$ and δ is sufficiently small. This completes the proof of (4.2) and thereby also the proof of Theorem 4.1.

We shall now consider the case of a general manifold M (without boundary). By Lemma 1.1, for any $x^0 \in M$ there is a suitable diffeomorphism of a neighborhood W of x^0 such that in the new coordinates $W \cap M$ has the form

(4.6) $$x_{k+1} = 0, \cdots, x_n = 0, \qquad x_1^2 + \cdots + x_k^2 \leq \delta^2 \quad (x^0 = 0)$$

and $a(x)$ satisfies (1.13), (1.14). Set $x = (x', x'')$, $x' = (x_1, \cdots, x_k)$, $x'' = (x_{k+1}, \cdots, x_n)$, $\alpha_{\lambda\mu}(x') = a_{k+\lambda, k+\mu}(x', 0)$ $(1 \leq \lambda, \mu \leq n-k)$.

Denote by $\alpha(x')$ the $(n-k)\times(n-k)$ matrix $(\alpha_{ij}(x'))$. If $d(x^0)=2$ and $n-k>2$, we introduce the $(n-k)\times(n-k)$ symmetric matrix $\alpha_\epsilon^0(x')=(\alpha_{ij}^0(x'))$ $(\epsilon>0)$, where

$$\alpha_{11}^0(x')=\alpha_{22}^0(x')=(1-\epsilon)\sum_{\lambda=3}^{n-k}\alpha_{\lambda\lambda}(x'),\qquad \alpha_{12}^0(x')=0,$$

$$\alpha_{1j}^0(x')=-2\alpha_{1j}(x'),\qquad \alpha_{2j}^0(x')=-2\alpha_{2j}(x')\ (3\le j\le n-k),$$

$$\alpha_{ii}(x')=2-\epsilon\ (3\le i\le n-k),\qquad \alpha_{ij}(x')=0\ (3\le i\le j\le n-k,\ i\ne j).$$

We shall require the condition:

(N_{x^0}) If $d(x^0)=2$ and $n-k>2$, then, for some $\epsilon>0$, the matrix $\alpha_\epsilon^0(x')$ is positive semidefinite for all $|x'|$ sufficiently small.

Definition. Let $n-k>2$. If at each point $x^0\in M$ where $d(x^0)=2$ the condition (N_{x^0}) holds, then we say that the condition (N) is satisfied.

Recall that $\alpha(x')$ is positive semidefinite. Hence $\alpha_{ij}^2\le\alpha_{ii}\alpha_{jj}$. It follows that, for any $\epsilon'>0$, $|\alpha_{ij}(x')|\le(1+\epsilon')\sqrt{\alpha_{jj}(x')}$ if $1\le i\le 2$, $3\le j\le n$. It is easily seen that if, for some $0<\theta<1/2$, $|\alpha_{ij}(x')|\le\theta\sqrt{\alpha_{jj}(x')}$ if $1\le i\le 2$, $3\le j\le n$, for all $|x'|$ sufficiently small, then the matrix $\alpha_\epsilon^0(x')$ is positive definite, for some $\epsilon>0$, provided $|x'|$ is sufficiently small; hence (N_{x^0}) follows in this case.

If $n-k=3$, the positivity of $\alpha(x')$ implies, for any $\epsilon'>0$, that $\alpha_{13}^2(x')+\alpha_{23}^2(x')\le(1+\epsilon')\alpha_{33}(x')$ provided $|x'|$ is sufficiently small. If $\alpha_{13}^2(x')+\alpha_{23}^2(x')\le(\tfrac{1}{2}-\epsilon_0)\alpha_{33}(x')$ for some $\epsilon_0>0$, then $\alpha_\epsilon^0(x')$ is positive definite for some $\epsilon>0$, provided $|x'|$ is sufficiently small; hence (N_{x^0}) follows in this case.

Theorem 4.2. *Let M be a k-dimensional C^2 submanifold of R^n $(0\le k\le n-1)$, and let* (A) *hold. Assume also that $a(x)$ is twice continuously differentiable in a neighborhood of M. If $d(x)\ge 2$ and if either $n-k=2$ or* (N) *holds, then* (1.3) *is satisfied, i.e., M is nonattainable.*

Proof. Consider first the case where M is bounded. Let $x^0\in M$ and let B_δ be a closed ball with center x^0 and radius δ. We wish to construct a function u in $B_\delta\setminus M$ such that

$$Lu(x)\le -c(d(x,M))^{\theta-2}\quad \text{if } x\in B_\delta\setminus M\ (c>0,\ 0<\theta<1), \tag{4.7}$$

$$|D_x u(x)|\le C/d(x,M)\quad \text{if } x\in B_\delta\setminus M, \tag{4.8}$$

$$u(x)\to\infty\quad \text{if } x\in B_\delta\setminus M,\ d(x,M)\to 0. \tag{4.9}$$

We first consider the case where $x^0 = 0$, $B_\delta \cap M$ is given by (4.6), and (when $n-k \geq 3$) (N_{x^0}) holds. If $d = d(0) \geq 3$ then we can construct u as in the proof of Theorem 3.1 (even with $\theta = 0$). We shall therefore consider only the case $d = 2$.

Let $m = n-k$, $x'' = (x_{k+1}, \cdots, x_n) = (y_1, \cdots, y_m)$ and introduce the distance function

$$r(x) = \left\{ \sum_{i,j=1}^{m} b_{ij}(x')y_i y_j \right\}^{1/2}, \quad b_{ij}(x') = b_{ji}(x'),$$

where the $b_{ij}(x')$ are still to be determined, and $b_{ij}(0) = \delta_{ij}$. Let $\phi(r)$ be the function defined by (4.5). We wish to determine the $b_{ij}(x')$ in such a way that the function $u(x) = \phi(r(x))$ satisfies (4.7)–(4.9), provided δ is sufficiently small.

Clearly,

$$\frac{\partial u}{\partial y_\lambda} = -\frac{1}{r^2}\left(\sum_{i=1}^{m} b_{i\lambda}y_i\right)e^{r^\theta/\theta},$$

$$\frac{\partial^2 u}{\partial y_\lambda \partial y_\mu} = \left[-\frac{1}{r^2} b_{\lambda\mu} + \frac{2}{r^4}\left(\sum_{i=1}^{m} b_{i\lambda}y_i\right)\left(\sum_{j=1}^{m} b_{j\mu}y_j\right) - \frac{r^\theta}{r^4}\left(\sum_{i=1}^{m} b_{i\lambda}y_i\right)\left(\sum_{j=1}^{m} b_{j\mu}y_j\right)\right] e^{r^\theta/\theta}.$$

Hence,

$$\sum_{\lambda,\mu=1}^{m} a_{\lambda\mu}\frac{\partial^2 u}{\partial x_\lambda \partial x_\mu} = \left[-\frac{1}{r^2}\sum_{\lambda,\mu=1}^{m} a_{\lambda\mu}b_{\lambda\mu} + \frac{2}{r^4}\sum_{\lambda,\mu=1}^{m} a_{\lambda\mu}\left(\sum_{i=1}^{m} b_{i\lambda}y_i\right)\left(\sum_{j=1}^{m} b_{j\mu}y_j\right) - \frac{r^\theta}{r^2}\sum_{\lambda,\mu=1}^{m} a_{\lambda\mu}\left(\sum_{i=1}^{m} b_{i\lambda}y_i\right)\left(\sum_{j=1}^{m} b_{j\mu}y_j\right)\right]e^{r^\theta/\theta}. \tag{4.10}$$

One is tempted to solve the system

$$F_{ij} \equiv b_{ij}\sum_{\lambda,\mu=1}^{m} a_{\lambda\mu}b_{\lambda\mu} - 2\sum_{\lambda,\mu=1}^{m} a_{\lambda\mu}b_{i\lambda}b_{j\mu} = 2(a_{ij}(0) - \delta_{ij})$$

in a neighborhood of $x' = 0$, $b_{ij} = \delta_{ij}$, in the form $b_{ij} = b_{ij}(x')$. Unfortunately, the Jacobian vanishes at the point where $x' = 0$, $b_{ij} = \delta_{ij}$. We therefore proceed differently. We define $b_{11} = a_{22}$, $b_{22} = a_{11}$, $b_{12} = -a_{12}$, $b_{jj} = 1$ if $3 \leq j \leq m$, $b_{ij} = 0$ if $1 \leq i \leq j$, $j \geq 3$, $i \neq j$. Set $A = \sum_{\lambda=3}^{m} a_{\lambda\lambda}$, in case $m \geq 3$. One can easily check that $F_{ij} = 0$ if $m = 2$ and $1 \leq i \leq j \leq 2$. If $m > 2$, then

$$F_{11} = \alpha_{22}A, \quad F_{22} = \alpha_{11}A, \quad F_{12} = -\alpha_{12}A,$$

$$F_{1j} = -2\alpha_{22}\alpha_{ij} + 2\alpha_{12}\alpha_{2j}, \quad F_{2j} = -2\alpha_{11}\alpha_{2j} + 2\alpha_{12}\alpha_{1j} \ (3 \leq j \leq m),$$

$$F_{jj} = \sum_{\lambda=1}^{m} \alpha_{\lambda\lambda}b_{\lambda\lambda} - 2\alpha_{jj} = 2 + O(|x'|) \quad \text{if } 3 \leq j \leq m,$$

$$F_{ij} = -2\alpha_{ij} \quad \text{if } 3 \leq i \leq j \leq m, \ i \neq j.$$

Suppose $m \geq 3$. Using the condition (N_{x^0}) we find that

$$\sum_{i,j=1}^{m} F_{ij}y_iy_j \geq \theta_0(y_3^2 + \cdots + y_m^2) \quad \text{for some } \theta_0 > 0,$$

provided δ is sufficiently small. Using this in (4.10), and noting that

$$-\sum_{\lambda,\mu=1}^{m} \alpha_{\lambda\mu}\left(\sum_{i=1}^{m} b_{i\lambda}y_i\right)\left(\sum_{j=1}^{m} b_{j\mu}y_j\right) = -y_1^2 - y_2^2 + \sum_{i,j=1}^{m} O(|x'|)y_iy_j,$$

we get

$$\sum_{\lambda,\mu=1}^{m} \alpha_{\lambda\mu}(x')\frac{\partial^2 u}{\partial x_\lambda \partial x_\mu} \leq \left[-\theta_0\frac{y_3^2 + \cdots + y_m^2}{r^4} - r^\theta\frac{y_1^2 + y_2^2}{r^4} + O(|x'|)\frac{r^\theta}{r^2}\right]e^{r^\theta/\theta}$$

(4.11)

$$\leq \left[-\theta_0\frac{r^\theta|y|^2}{r^4} + O(|x'|)\frac{r^\theta}{r^2}\right]e^{r^\theta/\theta} \leq -\frac{1}{2}\theta_0\frac{r^\theta}{r^2}$$

provided δ is sufficiently small. The final inequality is valid (by obvious modifications in the proof) also when $m = 2$.

Next, if $1 \leq l, h \leq k$, $1 \leq i \leq m$,

$$\frac{\partial r}{\partial x_l} = O(r), \quad \frac{\partial^2 r}{\partial x_l \partial x_h} = O(r), \quad \frac{\partial r}{\partial x_{k+i}} = O(1), \quad \frac{\partial^2 r}{\partial x_l \partial x_{k+i}} = O(1).$$

Hence,

$$\frac{\partial u}{\partial x_l} = O(1), \quad \frac{\partial^2 u}{\partial x_l \partial x_h} = O(1), \quad \frac{\partial u}{\partial x_{k+i}} = O\left(\frac{1}{r}\right), \quad \frac{\partial^2 u}{\partial x_l \partial x_{k+i}} = O\left(\frac{1}{r}\right).$$

Further,

$$\left|[a_{k+\lambda,k+\mu}(x', x'') - a_{k+\lambda,k+\mu}(x', 0)]\frac{\partial^2 u}{\partial x_{k+\lambda}\partial x_{k+\mu}}\right| \leq C|x''|\frac{C}{r^2} = \frac{C}{r}.$$

From (4.11) and the subsequent estimates it follows that

$$Lu \le -\tfrac{1}{2}\theta_0 r^\theta/r^2 + C/r \le -cr^\theta/r^2 \qquad (c > 0)$$

provided δ is sufficiently small. Thus (4.7) has been established. The assertions (4.8), (4.9) obviously hold.

Having established (4.7)–(4.9) in the special coordinates where $B_\delta \cap M$ is given by (4.6) and (1.13) holds, we can now return to the original coordinates, and conclude (cf. Lemma 1.1′):

For every $y \in M$ there is a ball $B(y, \delta_y)$ with center y and radius δ_y and a C^2 function $u^y(x)$ defined in $B(y, \delta_y)\setminus M$, such that

$$(4.12) \qquad Lu^y < -c(d(x, M))^{\theta-2} \quad \text{if } x \in B(y, \delta_y)\setminus M \ (c > 0),$$

$$(4.13) \qquad |D_x u^y(x)| \le C/d(x, M) \quad \text{if } x \in B(y, \delta_y)\setminus M,$$

$$(4.14) \qquad u^y(x) \to \infty \quad \text{if } x \in B(y, \delta_y)\setminus M,\ d(x, M) \to 0.$$

Cover a small neighborhood W of M by a finite number of balls $B(y, \delta_y)$. Denote these balls by $B_i = B(y_i, \delta_{y_i})$ and the corresponding functions $u^y(x)$ by $u^i(x)$; $1 \le i \le l$.

Let $\{\zeta_i\}$ be a partition of unity subordinate to the covering $\{B_i\}$, and set

$$u_i(x) = \begin{cases} \zeta_i u^i & \text{if } x \in B_i\setminus M, \\ 0 & \text{if } x \notin B_i. \end{cases}$$

Since $\zeta_i = 0$ outside B_i, $u_i(x)$ is in $C^2(W\setminus M)$. Further, by (4.12), (4.13),

$$Lu_i \le \zeta_i Lu^i + C/d(x, M) \le -c\zeta_i (d(x, M))^{\theta-2} + C/d(x, M)$$

if $x \in B_i\setminus M$. Setting $u = \Sigma_{i=1}^l u_i$, we get

$$Lu \le -c\sum_{i=1}^{l} \zeta_i (d(x, M))^{2-\theta} + \frac{C}{d(x, M)} < 0$$

if $x \in W\setminus M$ and $c(d(x, M))^{1-\theta} < 1/C$, since $\Sigma\zeta_i = 1$ on W.

From (4.14) we also have $u(x) = \Sigma_{i=1}^l \zeta_i(x) u^i(x) \to \infty$ if $x \in W\setminus M$, $d(x, M) \to 0$. An application of Lemma 2.1 with $\Omega = R^n$, $K = M$ now yields the assertion of Theorem 4.2, in case M is a bounded set.

Consider next the case where the set M is unbounded. We modify the above construction of u. Thus, instead of a finite covering of M by balls B_i, we now use a countable (but locally finite) covering. Further, the radii of the B_i may decrease to 0 as $i \to \infty$. However, there is still a neighborhood W of M such that $Lu(x) < 0$ if $x \in W \setminus M$, $u(x) \to \infty$ if $x \in W \setminus M$, $d(x, M) \to 0$; the last relation holds uniformly in x in bounded subsets. The "thickness" of $W \setminus M$ may go to zero at ∞.

Now, if the assertion (1.3) is false, then there is an event G with $P_x(G) > 0$ such that, if $\omega \in G$, $\xi(t, \omega) \in M$ for some $t = t_\omega < \infty$. Introduce the balls $B_m = \{y; |y| < m\}$, m a positive integer, and the events

$$G_m = \{\omega \in G; \xi(t, \omega) \in B_m \text{ if } 0 \le t \le t_\omega\}.$$

Clearly $G = \bigcup_{m=1}^{\infty} G_m$. Hence there is an m for which $P_x(G_m) > 0$. But this contradicts Lemma 2.1 in the case where $K = M \cap \overline{B}_m$, $\Omega = B_m$.

Remark. Let M be a manifold with boundary. Suppose that $d(x) \ge 3$ if $x \in \partial M$ and $d(x) \ge 2$ and (when $n - k \ge 3$) (N_x) holds for each $x \in M$. Then M is nonattainable. Indeed, if $x^0 \in \partial M$ then we can construct a function satisfying (4.12)–(4.14) by the proof of Theorem 3.1. If $x^0 \in M \setminus \partial M$, then we can construct u satisfying (4.12)–(4.14) as in the proof of Theorem 4.2. Now use partition of unity (as in the proof of Theorem 4.2) in order to complete the proof.

5. M consists of one point and $d = 1$. We shall consider primarily the case where M consists of one point x^0, and $x^0 = 0$. We begin, for simplicity, with the case $n = 2$. Without loss of generality we may take $a_{11}(0, 0) > 0$, $a_{22}(0, 0) = 0$. Since $a_{22}(x, y) \ge 0$, we conclude that $\partial a_{22}/\partial x = 0$, $\partial a_{22}/\partial y = 0$ at the origin. Hence, if $a_{22}(x, y)$ is in C^2 in a neighborhood of the origin, $a_{22}(x, y) = O(r^2)$ where $r^2 = x^2 + y^2$. From the inequality $|a_{12}| \le \sqrt{a_{11}}\,\sqrt{a_{22}}$ we see that $a_{12}(0, 0) = 0$. Hence, if $a_{12}(x, y)$ is continuously differentiable and $a_{22}(x, x)$ is twice continuously differentiable in a neighborhood of the origin, then

$$a(x, y) = \begin{pmatrix} A + o(1) & Mx + Ny + o(r) \\ Mx + Ny + o(r) & Bx^2 + Cxy + Dy^2 + o(r^2) \end{pmatrix}, \quad A > 0, \tag{5.1}$$

as $r \to 0$. Since the matrix $a(x, y)$ is positive semidefinite, $B \ge 0$, $D \ge 0$, $M^2 \le AB$, $C^2 \le 4BD$.

We shall assume:

$$B > 0, \tag{5.2}$$

and $|C|$, $|M|$ are "sufficiently small," so that for some $p > 1$, $q > 1$, $p' > 1$, $q' > 1$, $p_0 > 1$, $q_0 > 1$, where $1/p + 1/q = 1$, $1/p' + 1/q' = 1$, $1/p_0 + 1/q_0 = 1$, and for some $\lambda > 0$, the following inequalities hold.

(5.3) $$|C|\lambda/p + 2|M|/p' < B\lambda,$$

(5.4) $$|C|/q \le D,$$

(5.5) $$|M|\lambda/q' < 2A,$$

(5.6) $$4|M|\lambda/q_0 + 2A < B\lambda,$$

(5.7) $$4|M|/p_0 + B\lambda < 6A.$$

Finally, we assume:

(5.8) $$\text{If } D = 0, \text{ then } a_{22}(x, y) = Bx^2(1 + o(1)).$$

Notice that if $|M|$ is sufficiently small so that $4|M|/q_0 < B$, $2|M|/p_0 < 3A$ and

$$\alpha' \equiv \frac{2A}{B - 4|M|/q_0} < \frac{2(3A - 2|M|/p_0)}{B} \equiv \alpha'',$$

then any λ satisfying $\alpha' < \lambda < \alpha''$ also satisfies (5.6), (5.7).

Regarding the b_i, we require that $b_2(0, 0) = 0$. Hence, if $b_2(x, y)$ is continuously differentiable in a neighborhood of the origin, then

(5.9) $$b_2(x, y) = c_1 x + c_2 y + o(r).$$

Theorem 5.1. *Let* (5.1)–(5.9) *hold. Then, for any* $(x, y) \neq (0, 0)$,

(5.10) $$P_{(x,y)}\{|\xi(t)| = 0 \text{ for some } t > 0\} = 0.$$

Proof. Let $R(x, y) = x^4 + \mu x^2 y^2 + \lambda y^2$ where λ is a positive number satisfying (5.3)–(5.7), and μ is a positive constant to be determined later on. We shall find a function $u = \Phi(R)$ such that, for some small $\gamma > 0$,

(5.11) $$L\Phi(R) \le 0 \quad \text{if } 0 < R < \gamma,$$

(5.12) $$\Phi(R) \to \infty \quad \text{if } R \to 0.$$

By Lemma 2.1, this will complete the proof of the theorem.

We can write Lu in the form $Lu = \alpha\Phi''(R) + \beta\Phi'(R)$. If we show that

(5.13) $$\alpha \ge 0, \quad \beta \ge \alpha/R,$$

(5.14) $$\Phi''(R) + \Phi'(R)/R = 0, \quad \Phi'(R) < 0,$$

then (5.11) follows. A solution of (5.14) is given by $\Phi(R) = \log(1/R)$. With this $\Phi(R)$, (5.12) is also satisfied. Thus, it remains to verify (5.13).

We shall use the following notation: if E is a constant then $\hat{E}$ is a function of the form $E(1 + o(1))$.

Now, by direct calculation one finds that

$$\alpha = 16\hat{A}x^6 + 4\hat{B}\lambda^2x^2y^2 + 4\hat{D}\lambda^2y^4 + 4\hat{C}\lambda^2xy^3 + 8M\lambda x^4y,$$

$$\beta R = (12\hat{A} + 2\hat{B}\lambda)x^6 + (12\hat{A}\lambda + 2\hat{B}\lambda^2)x^2y^2 + (2D\lambda^2 + 2\hat{A}\lambda\mu + 2c_2\lambda^2)y^4$$
$$+ 2C\lambda^2xy^3 + 2c_1\lambda^2xy^3.$$

Using the inequalities

$$|xy^3| \le x^2y^2/p + y^4/q, \qquad |x^4y| \le x^2y^2/p' + x^6/q'$$

and (5.3)–(5.5), we find that $\alpha \ge 0$ (if $D = 0$ we use also (5.8)).

In order to show that $\beta R \ge \alpha$, we use the inequalities

$$|xy^3| \le \eta x^2y^2 + y^4/4\eta, \qquad |x^4y| \le x^2y^2/p_0 + x^6/q_0,$$

in both α and βR. We then obtain the inequality

$$\beta R - \alpha \ge \hat{\gamma}_1x^6 + \hat{\gamma}_2x^2y^2 + \hat{\gamma}_3y^4 \quad (\hat{\gamma}_i = \gamma_i(1 + o(1))).$$

By (5.6), $\gamma_1 > 0$, and by (5.7), $\gamma_2 > 0$ provided η is sufficiently small. Since μ does not appear in γ_1, γ_2, and since it appears only in the additive term $2\hat{A}\lambda\mu$ of γ_3, we can choose μ so large that $\gamma_3 > 0$. It follows that $\beta R \ge \alpha$. We have thus completed the proof of (5.13).

Remark 1. When $n = 1$ and $d(x^0) = 1$ then $\sigma(x^0)$ is nondegenerate (i.e., $\sigma(x^0) \neq 0$), and (by [6], for instance)

$$P_x\{\xi(t) = x^0 \text{ for some } t > 0\} \to 1 \quad \text{if } x \to x^0.$$

Remark 2. The condition (5.2) is essential for the validity of the assertion of Theorem 5.1. Consider, for example, the system

$$d\xi_1 = dw_1, \qquad d\xi_2 = \sigma(\xi_1, \xi_2)dw_2,$$

where $\sigma(x_1, 0) = 0$. If $(\xi_1(0), \xi_2(0)) = (\alpha, 0)$, then the solution is $\xi_1(t) = \alpha + w_1(t)$, $\xi_2(t) = 0$. Hence

$$P_{(\alpha,0)}\{|\xi(t)| = 0 \text{ for some } t > 0\} = 1.$$

Remark 3. A quick review of the proof of (5.13) shows that we have actually proved also that $\beta \ge (1 + \delta)\alpha/R$ for some sufficiently small $\delta > 0$. Hence we can take in the above proof $\Phi(R) = 1/R^\delta$.

Consider now the case $n \ge 2$. Without loss of generality we may assume that $a_{11}(0) > 0$, $a_{ii}(0) = 0$ if $2 \le i \le n$. If $a_{ii}(x)$ $(2 \le i \le n)$ is in C^2 in a neighborhood of 0 then $a_{ii}(x) = O(|x|^2)$. It follows that $a_{1i}(x) = O(|x|)$, $a_{ij}(x) = O(|x|^2)$ $(2 \le i, j \le n)$.

Setting $y_j = x_{j+1}$ $(1 \leq j \leq n-1)$, $m = n-1$, and assuming that the a_{ij} are in C^2 in a neighborhood of the origin, we then have:

$$
(5.15)\quad
\begin{aligned}
a_{11} &= A + o(1),\ A > 0,\\
a_{1j} &= M_j x_1 + \sum_{l=1}^{m} M_{jl} y_l + o(r) \quad (2 \leq j \leq n),\\
a_{jj} &= B_j x_1^2 + \sum_{l=1}^{m} C_{jl} x_1 y_l + \sum_{l,k=1}^{m} D_{j,lk} y_l y_k + o(r^2) \quad (2 \leq j \leq n),\\
a_{ij} &= E_{ij} x_1^2 + \sum_{l=1}^{m} E_{ij,l} x_1 y_l + \sum_{l,k=1}^{m} E_{ij,lk} y_l y_k + o(r^2) \quad (2 \leq i, j \leq n).
\end{aligned}
$$

We shall assume:

$$(5.16)\qquad \sum_{j=2}^{n} B_j > 0,$$

$$(5.17)\qquad \sum_{l,k,i,j} (D_{i,lk}\delta_{ij} + E_{ij,lk}) y_l y_k y_i y_j \geq c|y|^4 \quad (c > 0),$$

(5.18) $|C_{jl}|$, $|M_j|$, $|E_{ij}|$, $|E_{ij,k}|$ are sufficiently small.

Notice that the left-hand side of (5.17) is always ≥ 0. In case (5.17) does not hold, we shall have to impose further restrictions.

(5.19) If $c = 0$ in (5.17) then $C_{jl} = 0$, $E_{ij,k} = 0$ and the terms $o(r)$, $o(r^2)$ occurring in a_{jj}, a_{ij} (in (5.15)) are replaced by $o(|x_1|)$ and $o(x_1^2)$ respectively.

Theorem 5.2. *Let* (5.15), (5.16) *hold. Assume also that either* (5.17), (5.18), *hold, or* (5.19) *holds and the* $|M_j|$, $|E_{ij}|$ *are sufficiently small. Then,*

$$(5.20)\qquad P_x\{\xi(t) = 0 \text{ for some } t > 0\} = 0 \quad \text{if } x \neq 0.$$

The proof is similar to the proof of Theorem 5.1. We now take $u = \Phi(R)$ with Φ as before, but with

$$R(x) = x_1^4 + \mu \sum_{j=1}^{m} x_1^2 y_j^2 + \lambda \sum_{j=1}^{m} y_j^2;$$

λ is a suitable positive number and μ is a sufficiently large positive number.

Theorems 5.1, 5.2 can be extended to manifolds M of special form. As a trivial example, take $n = 3$, M the z-axis, and a_{ij} for $1 \leq i, j \leq 2$ as in Theorem

5.1. Then $d = 1$ along M, and M is nonattainable. The proof uses the same function $u = \Phi(R)$ as in the proof of Theorem 5.1.

6. The case $d(x) = 0$. The following theorem was proved by Friedman and Pinsky [4].

Theorem 6.1. *Let G be a closed bounded domain in R^n with C^3 boundary M, and denote by $\nu = (\nu_1, \cdots, \nu_n)$ the outward normal to G at M. Let* (A) *hold, and assume that*

$$\sum_{i,j=1}^{n} a_{ij}\nu_i\nu_j = 0 \quad \text{on } M, \tag{6.1}$$

$$\langle b, \nu\rangle + \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 \rho}{\partial x_i \partial x_j} \geq 0 \quad \text{on } M, \tag{6.2}$$

where $\rho(x) = \operatorname{dist}(x, M)$ *if* $x \notin \operatorname{int} G$. *Then, for any* $x \notin G$,

$$P_x\{\xi(t) \in M \text{ for some } t > 0\} = 0. \tag{6.3}$$

If the inequality in (6.2) is reversed at one point x_0 of M, then the assertion (6.3) is false; in fact (cf. [4]),

$$P_x\{\xi(t) \in M \text{ for some } t > 0\} \to 1 \quad \text{if } x \notin G,\ x \to x^0.$$

Notice that the condition (6.1) means that $d(x) = 0$ along M.

We also note (see [4]) that, when the a_{ij} belong to C^1 in a neighborhood of M, the condition (6.2) is equivalent to:

$$\sum_{i=1}^{n}\left(b_i - \sum_{j=1}^{n}\frac{\partial a_{ij}}{\partial x_j}\right)\nu_i \geq 0 \text{ on } M. \tag{6.4}$$

The proof of Theorem 6.1 follows by producing a function u satisfying $Lu \leq \mu u$ in a $\hat{G}$-neighborhood of M, $\hat{G} = R^n \setminus G$, $\mu > 0$, $u(x) \to \infty$ if $x \in \hat{G}$, $\rho(x) \to 0$. Such a function is

$$u(x) = 1/(\rho(x))^\epsilon \quad \text{for any } \epsilon > 0. \tag{6.5}$$

Suppose now that G is a bounded, closed and convex domain, with piecewise C^3 boundary. Thus each point x of the boundary M lies on a finite number of C^3 $(n-1)$-dimensional submanifolds of M, say $M_{i_1}, \cdots, M_{i_s}$. Their intersection is a k-dimensional C^3 manifold through x $(k = n - s)$. Denote by N_x the $(n-k)$-dimensional space of the normals to this submanifold at x.

The function $D_x\rho(x)$ is continuous. On the other hand, $D_x^2\rho(x)$ is piecewise continuous; denote by Σ the set of its discontinuities.

Theorem 6.1 extends to the present case provided (6.1) holds for any $x \in M$, $\nu \in N_x$, and provided (6.2) is replaced by

$$(6.6) \qquad \varliminf_{y \to x} \frac{1}{\rho(y)} \left[\sum_{i=1}^{n} b_i(y) \frac{\partial}{\partial y_i} \rho(y) + \sum_{i,j=1}^{n} a_{ij}(y) \frac{\partial^2}{\partial y_i \partial y_j} \rho(y) \right] \geq -C$$

($y \notin G \cup \Sigma$, C positive constant).

Notice that condition (6.1) for all $\nu \in N_x$ can be interpreted as $d_{M^\perp}(x) = 0$, when the notion of $d_{M^\perp}$ is extended in a natural way to the case of a piecewise smooth manifold.

When dim $N_x = n$, the conditions (6.1) for all $\nu \in N_x$ and (6.6) reduce to $\sigma(x) = 0$, $b(x) = 0$.

Suppose next that M is a piecewise C^3 bounded submanifold in R^n, of any dimension k $(1 \leq k \leq n-1)$, with piecewise C^3 boundary ∂M. We can still extend Theorem 6.1 (taking $u(x) = 1/(d(x, M))^\epsilon$, $\epsilon > 0$) provided the following conditions hold:

(i) $d(x, M)$ is continuously differentiable and its second derivative is piecewise continuous, in some $\hat{M}$-neighborhood of M; $\hat{M} = R^n \setminus M$; denote by $\hat{\Sigma}$ the set of discontinuities of $D_x^2 d(x, M)$ in $\hat{M}$.

(ii) For any $x \in \operatorname{int} M$, (6.1) holds for all $\nu \in N_x$ (N_x is the space of normals to M at x), and

$$(6.7) \quad \varliminf_{y \to x} \frac{1}{d(y, M)} \left[\sum_{i=1}^{n} b_i(y) \frac{\partial}{\partial y_i} d(y, M) + \sum_{i,j=1}^{n} a_{ij}(y) \frac{\partial^2}{\partial y_i \partial y_j} d(y, M) \right] \geq -C$$

($y \notin M \cup \hat{\Sigma}$, C positive constant) uniformly with respect to x.

(iii) For any $x \in \partial M$, (6.1) holds for all ν normal to ∂M at x, and (6.7) holds.

Remark 1. In [4], Theorem 6.1 was extended to G convex with piecewise C^3 boundary under the assumption that $a_{ij} \in C^2$. Using Lemma 2.1, we see that this assumption is not needed.

Remark 2. In the proof of Lemma 1.1, we had to use Itô's formula for a C^1 function u with piecewise continuous second derivatives, satisfying $Lu \leq \mu u$. We extended u into U and mollified U into U_λ; then applied Itô's formula to U_λ, and finally let $\lambda \to 0$. The same procedure can be carried out in the proof of Theorem 2.2 in [4]. This simplifies that proof and also enables us to eliminate the restriction (D) (made in that theorem).

7. Mixed case: an application to the Dirichlet problem. Set $d(x) = r_{M^\perp}(x)$, $d'(x) = r_{(\partial M)^\perp}(x)$. We shall consider the case where $n = 2$, M is an arc, and

$$(7.1) \qquad d(x) = 0 \quad \text{if } x \in M, \qquad d(x) = 1 \quad \text{if } x \in \partial M.$$

One can also consider, by the same method, other mixed cases. The motivation for studying the particular case (7.1) arises from an application to the Dirichlet problem; this will be considered later on.

The idea for handling the mixed case (7.1) is to form two functions, u_1 and u_2, such that:

(i) u_1 is a function constructed for the case $d(x) = 0$ (in §6);

(ii) u_2 is a function constructed for the case $d'(x) = 1$ (in §5);

(iii) u_1 and u_2 fit together in a continuously differentiable manner.

For simplicity we shall deal primarily with the case:

$$M = \{(x_1, x_2);\ x_1 = 0,\ 0 \le x_2 \le \beta\}. \tag{7.2}$$

The case of a general arc M follows by first performing a local diffeomorphism, mapping the arc onto a linear segment as in (7.2).

Let Ω be a bounded closed domain lying in the half-plane $x_1 \ge 0$, with boundary $\partial_1\Omega \cup \partial_2\Omega$, where $\partial_1\Omega = \{(x_1, x_2); -\alpha \le x_1 \le \alpha, x_2 = 0\}$ and $\partial_2\Omega$ lies in the half-plane $x_2 > 0$. We assume that $M \subset \Omega$.

The stochastic differential system is

$$d\xi_i = \sum_{s=1}^{2} \sigma_{is}(\xi)\, dw_s + b_i(\xi)\, dt \qquad (i = 1, 2). \tag{7.3}$$

Denote by τ the exit time from Ω. In view of the application for the Dirichlet problem, we are interested in the process $\xi(t)$ only as long as $t < \tau$. Thus, we would like to prove that M is nonattainable in time $< \tau$, i.e.,

$$P_x\{\xi(t) \in M \text{ for some } t < \tau\} = 0 \quad \text{if } x \in \Omega\backslash M. \tag{7.4}$$

First we assume that (6.1), (6.4) hold with respect to both sides of M, i.e., if $a = \sigma\sigma^*/2$ then

$$a_{11}(0, x_2) = 0 \quad \text{for } 0 \le x_2 \le \beta, \tag{7.5}$$

$$b_1(0, x_2) - \frac{\partial a_{11}(0, x_2)}{\partial x_1} - \frac{\partial a_{12}(0, x_2)}{\partial x_2} = 0 \quad \text{if } 0 \le x_2 \le \beta. \tag{7.6}$$

If the point $(0, \beta)$ lies on the boundary of Ω, then (7.4) follows from the proof of Theorem 6.1 (when slightly modified). Recall that we apply here Lemma 2.1 with any function

$$u(x) = c/(x_1^2)^\epsilon \qquad (c > 0,\ \epsilon > 0). \tag{7.7}$$

In applications, however, β may be small, so that

$$(0, \beta) \in \text{int } \Omega. \tag{7.8}$$

We shall henceforth assume that (7.8) holds, and that not all the $a_{ij}(0, \beta)$ $(1 \leq i, j \leq 2)$ vanish. (If they all vanish then (7.4) again follows from the results of §6.)

Assuming the a_{ij} to be in C^2 in a neighborhood of $(0, \beta)$, and recalling (7.5), we then have:

$$a(x_1, x_2) = \begin{pmatrix} Bx_1^2 + Cx_1(x_2 - \beta) + D(x_2 - \beta)^2 + o(r^2) & Mx_1 + N(x_2 - \beta) + o(r) \\ Mx_1 + N(x_2 - \beta) + o(r) & A + o(1) \end{pmatrix}, \tag{7.9}$$

$$A > 0,$$

where $r^2 = x_1^2 + (x_2 - \beta)^2$. We shall require (cf. (5.9)) that

$$b_1(x_1, x_2) = c_1 x_1 + c_2(x_2 - \beta) + o(r). \tag{7.10}$$

From (7.10), (7.6) it follows that $N = 0$ in (7.9). We finally require that either

$$D > 0,\ B > 0,\ |C| \text{ is sufficiently small,} \tag{7.11}$$

or

$$D > 0,\ B = 0,\ C = 0, \text{ and } a_{11}(x_1, x_2) = Bx_1^2(1 + o(1)). \tag{7.12}$$

Consider the function

$$u(x) = 1/(R(x))^{\delta} \qquad (\delta > 0), \tag{7.13}$$

where $R(x) = (x_2 - \beta)^4 + \mu(x_2 - \beta)^2 x_1^2 + \lambda x_1^2$. By Remark 3 at the end of the proof of Theorem 5.1, $Lu \leq 0$ if $0 < x_1^2 + (x_2 - \beta)^2 < \epsilon_0$ for some $\epsilon_0 > 0$, provided δ is sufficiently small; here μ, λ are suitable positive constants.

Note that the function

$$d(x) = \begin{cases} R(x) & \text{if } x_2 > \beta, \\ \lambda x_1^2 & \text{if } x_2 < \beta \end{cases}$$

is C^1 and piecewise C^2. Recalling (7.7), (7.13), we conclude that the function $u(x) = 1/(d(x))^{\delta}$ is C^1 and piecewise C^2 in $\Omega \setminus M$, and $Lu \leq 0$ for x in $(\Omega \setminus M)$-neighborhood of M, $x_2 \neq \beta$, $u(x) \to \infty$ if $x \in \Omega \setminus M$, $d(x, M) \to 0$. Hence, by Lemma 2.1, (7.4) holds. We sum up:

Theorem 7.1. *Let* (7.5), (7.6), (7.9), (7.10) *hold, and let* (7.11) *or* (7.12) *hold. Then* (7.4) *is satisfied.*

An application. In [5] Friedman and Pinsky have considered the Dirichlet problem for $n = 2$,

$$Lu = 0 \text{ in a bounded domain } G,$$

$$u = f \text{ on } \Sigma_2 \cup \Sigma_3,$$

$$u(x) \to f_i^+ \text{ if } x \to p_i, \quad x \in N_i^+ \quad (1 \le i \le l),$$

$$u(x) \to f_i^- \text{ if } x \to p_i, \quad x \in N_i^- \quad (1 \le i \le l);$$

f is a given continuous function, and the f_i^+, f_i^- are given numbers. Here Σ_3 is the part of the boundary where $\Sigma\, a_{ij}\nu_i\nu_j > 0$; Σ_2 is the part where (6.1) holds and

$$\sum_{i=1}^{2}\left(b_i - \sum_{j=1}^{2} \frac{\partial a_{ij}}{\partial x_j}\right)\nu_i > 0 \quad (\nu \text{ outward normal})$$

and Σ_3 is the remaining boundary. The points p_i lie in Σ_1, $N_i^+ \cap N_i^- = \varnothing$, and the closure of $N_i^+ \cup N_i^-$ constitutes a $\bar{G}$-neighborhood of p_i. The open sets N_i^+, N_i^- have a common boundary $\hat{\Delta}_i$. The location of the points p_i can be determined explicitly from the a_{ij}, b_i. One of the assumptions made in [5] is that there is a curve Δ_i, initiating at p_i and terminating on the boundary of G, such that Δ_i is an extension of $\hat{\Delta}_i$ and such that the conditions (6.1), (6.2) (with $n = 2$) hold along Δ_i, from both sides of it. The curve Δ_i is called a "boundary spoke". The rank of $a(x)$ for any $x \in \Delta_i \cap G$ is 1.

The existence of such a curve Δ_i was assumed in order to assert that $\xi(t)$ does not cross $\hat{\Delta}_i$ if t is sufficiently large, and $t < \tau$.

Using Theorem 7.1, we can now replace the assumption regarding the "boundary spoke" Δ_i by the following set of assumptions:

(a) There is a curve $\tilde{\Delta}_i$ initiating at p_i, containing $\hat{\Delta}_i$ and lying in G (except for its initial point p_i); denote its end point by q_i.

(b) The conditions (6.1), (6.2) (with $n = 2$) hold along $\tilde{\Delta}_i$, from both sides of it.

(c) The conditions "analogous" to (7.9), (7.10) and either (7.11) or (7.12) hold.

By the "analogous" conditions we mean the following:

Perform a local diffeomorphism in a neighborhood W of $\tilde{\Delta}_i$ which maps $\tilde{\Delta}_i$ onto the line segment (7.2) and $W \cap \partial G$ onto a segment $\{(x_1, 0); -\alpha < x_1 < \alpha\}$. Then, the transformed a_{ij}, b_i satisfy (7.9), (7.10) and either (7.11) or (7.12).

We sum up:

The "long" "boundary spoke" Δ_i going from p_i to the boundary of G can be replaced by a "short" "boundary spoke" $\tilde{\Delta}_i$ going from p_i to a point q_i in G, provided the condition (c) *is satisfied at q_i.*

REFERENCES

1. A. Bonami, N. Karoui, B. Roynette and H. Reinhard, *Processus de diffusion associé à un opérateur elliptique dégénéré*, Ann. Inst. H. Poincaré Sect. B (N. S.) 7 (1971), 31–80. MR 44 #7637.

2. A. Friedman, *Partial differential equations*, Holt, Rinehart and Winston, New York, 1969.

3. A. Friedman and M. A. Pinsky, *Asymptotic behavior of solutions of linear stochastic differential systems*, Trans. Amer. Math. Soc. 181 (1973), 1–22.

4. ———, *Asymptotic stability and spiraling properties for solutions of stochastic equations*, Trans. Amer. Math. Soc. 186 (1973), 331–358.

5. ———, *Dirichlet problem for degenerate elliptic equations*, Trans. Amer. Math. Soc. 186 (1973), 359–383.

6. I. I. Gihman and A. V. Skorohod, *Stochastic differential equations*, Naukova Dumka, Kiev, 1968. (Russian). MR 41 #7777.

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